Role of helicity in turbulence (Geometric Aspect of Partial Differential Equations and Conservation Laws)

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Role of helicity in turbulence

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The energy is a conservative quantity of the inviscid (dissipationless) fluid, which plays the most important role in the theory of turbulence although turbulence is a dissipative system. The helicity is another inviscid conservative quantity, which is defined as

$$H=\int_{D}\boldsymbol{u}\cdot\boldsymbol{\omega}\,d\boldsymbol{x}. \tag{1}$$

Here D is the domain where the fluid exists and we assume that there is no helicity input or output on the boundary (such condition holds for the periodic boundary and $\mathbb{R}^{3}$). The $\boldsymbol{u}$ and $\boldsymbol{\omega}=\nabla\times\boldsymbol{u}$ are the velocity and the vorticity. The conservation of the helicity was found rather recently in 1960's and its roles has not been explored perhaps due to its lack of sign definiteness. We discuss here possible roles of the helicity in the theory of turbulence [1].

We start to explain what we mean by the important role of the energy. The central outcome of a turbulence theory concerns the exponent α of the turbulent velocity increment[2]:

$$u(x+r,t)-u(x,t)\propto r^{\alpha}, \tag{2}$$

where we ignore the vectorial property of the velocity $\boldsymbol{u}$, the spatial coordinate $\boldsymbol{x}$ and $\boldsymbol{r}$ for simplicity. Furthermore we assume the turbulence reaches statistically steady state. In 1941 Kolmogorov predicts that $\alpha=1/3$, namely

$$u(x+r,t)-u(x,t)\propto r^{1/3}, \tag{3}$$

based on the fact that the nonlinearity of the Navier-Stokes equations conserves the energy [2]. There is of course the viscous term, which decrease the energy, in the incompressible Navier-Stokes equations

$$\partial_{t}\boldsymbol{u}+(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}=-\frac{1}{\rho_{0}}\nabla p+\nu\nabla^{2}\boldsymbol{u},\quad \nabla\cdot\boldsymbol{u}=0. \tag{4}$$

However in the well-developed turbulence, the viscous term is smaller by several orders of magnitude than the nonlinear term in a suitable range of spatial scale r, where the law Eq.(2) holds. So we focus on the nonlinearity only. In any case, the prediction Eq.(3) has been confirmed to be true by various experiments and computer simulations of turbulence.

How one can get Eq.(3)? The nonlinearity of the Navier-Stokes equations actually distributes the energy among all the modes[3]. The first assumption is that the distribution occurs in a simple manner: the energy flux [4] associated with the wavevector $k=|\boldsymbol{k}|$ becomes constant and independent of k. The second assumption is that the statistics of the turbulence velocity is determined by the constant energy flux ϵ. The velocity has the dimension of [Length / Time]. The energy flux ϵ has the dimension [Energy / Time] = [Length^{2}/Time^{3}]. We now make the dimension of the velocity (the left hand side of Eq.(3)) by using ϵ and r. The only dimensionally consistent combination is

$$u(x+r,t)-u(x,t)=C\epsilon^{1/3}r^{1/3}, \tag{5}$$

where C is a non-dimensional constant. This is a way to get the Kolmogorov law Eq.(3).

The above argument can be easily applied to other conservative quantities to obtain a possible exponent α in Eq.(2). So why not the helicity. The nonlinearity indeed does not decrease or increase the helicity but distribute it among all the modes. By assuming the helicity flux η, whose dimension is [Length / Time^{3}], becomes constant and independent of the length scale, we get

$$u(x+r,t)-u(x,t)=C'\eta^{1/3}r^{2/3}. \tag{6}$$

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[2] The velocity increment in th left hand side is understood as a suitable (spatial [average in x], temporal [average in t] or ensemble) average

[3] Let us imagine we have the Fourier transforms of the turbulent velocity and we call each Fourier transform $\hat{\boldsymbol{u}}(\boldsymbol{k},t)$ of the wavevector $\boldsymbol{k}$ as "modes".

[4] This energy flux is a sort of the area integral of the energy transfer vector dotted the unit normal vector $\boldsymbol{k}/|\boldsymbol{k}|$ on the sphere of $k=|\boldsymbol{k}|$ in the wavenumber space. For the precise definition, see Sec.6.2.2 and 6.2.3 of Ref.[2].

The question is whether or not the law Eq.(6) can be observed in experiments or simulations. The answer is no so far. Nevertheless the assumption, the helicity flux η becoming constant, is validated by numerical simulations. A reasonable explanation is that there is no pure helicity-dominated turbulence leading to Eq.(6) since the energy and the helicity are simultaneously transfered (so-called joint cascade) by the nonlinearity [5]. Hence in reality we can expect that

$$u(x+r,t)-u(x,t)=C''\epsilon^{\beta}\eta^{\gamma}r^{\delta}. \tag{7}$$

But now the exponents are not dimensionally determined uniquely.

Or it may be that the naive velocity difference in the left hand sides should be modified to observe the helicity-related scaling law Eq.(6). We need to come back to vectorial properties of the velocity field now. In fact the left hand side of the Kolmogorov law (5) holds for the velocity component u_L as

$$u_L(\boldsymbol{x}+\boldsymbol{r},t)-u_L(\boldsymbol{x},t)=C\epsilon^{1/3}|\boldsymbol{r}|^{1/3}, \tag{8}$$

where

$$u_L(\boldsymbol{x},t)=\boldsymbol{u}(\boldsymbol{x},t)\cdot\frac{\boldsymbol{r}}{|\boldsymbol{r}|},\quad \boldsymbol{u}_T(\boldsymbol{x},t)=\boldsymbol{u}(\boldsymbol{x},t)-\boldsymbol{u}(\boldsymbol{x},t)\cdot\frac{\boldsymbol{r}}{|\boldsymbol{r}|}. \tag{9}$$

Probably a helicity-flux related scaling law such as Eq.(6) or (7) holds among suitable components of $\boldsymbol{u}_T$. This may be seen by analyzing the isotropic form of the third order correlation functions of the velocity field. Certainly a simple dimensional analysis à la Kolmogorov is not able to capture this kind of subtle geometrical structure. If the helicity or its related quantity is able to capture a geometrical structure otherwise impossible, it can be a useful quantity to rectify geometrical properties of the turbulent velocity or vorticity field.

It has been known for a long time that turbulence spontaneously generates conspicuous geometrical structures like vortex sheets and tubes, whose dynamical origin and impact on statistics are not well understood [2]. Several properties of helicity suggest its usefulness. Firstly, the helicity density $\boldsymbol{u}\cdot\boldsymbol{\omega}$ measures how an instantaneous streamline (an integral curve of the velocity) is close to a right-hand screw (favorable in positive helicity) or a left-hand one (favorable in negative helicity). Secondly, in the case of idealized vortex lines, the helicity measures knottedness of the vortex lines [1]. Lastly, change in the helicity may be associated to a certain event which is called vorticity reconnection. With the presence of the viscosity, the vortex lines can touch and re-connect (without the viscosity, they cannot). When this topological change of vortex lines occurs, the viscous term (the last term) in the helicity budget equation

$$\begin{aligned}\frac{d}{dt}\left[\int \boldsymbol{u}\cdot\boldsymbol{\omega}\,d\boldsymbol{x}\right]+\int\nabla\cdot[\boldsymbol{u}\,(\boldsymbol{u}\cdot\boldsymbol{\omega})]\,d\boldsymbol{x}&=-\frac{1}{\rho_0}\int\nabla\cdot(p\,\boldsymbol{\omega})d\boldsymbol{x}+\int\nabla\cdot\left[\boldsymbol{\omega}\frac{|\boldsymbol{u}|^2}{2}\right]d\boldsymbol{x}\\ &-\nu\left[\int\nabla^2(\boldsymbol{u}\cdot\boldsymbol{\omega})d\boldsymbol{x}+\int\sum_{i,j=1}^{3}\left(\frac{\partial u_i}{\partial x_j}\right)\left(\frac{\partial \omega_i}{\partial x_j}\right)d\boldsymbol{x}\right].\end{aligned} \tag{10}$$

can generate (not dissipate) the helicity [3] [6]. We believe that the helicity is a good starting point to understand the relation between turbulence statistical law like Eq.(2) and dynamical events happening in turbulence.

References

[1] For authoritative reviews on helicity, see e.g., H.K. Moffatt and A. Tsinober, Helicity in laminar and turbulent flow, *Annual Reviews of Fluid Mechanics*, **24**, 281–312 (1992).

[2] See e.g., U. Frisch, *Turbulence: the legacy of A.N. Kolmogorov*, (Cambridge university press 1996).

[3] H. Aref and I. Zawadzki, Linking of vortex rings, *Nature*, **354**, 50–53 (1991).

[5] On the contrary there can be a pure energy-dominated turbulence, where the helicity is zero and the helicity flux is negligibly small.

[6] This means that the absolute value of the helicity can increase (as opposed to the energy can decrease) with dissipation but we do not call it "増エシティ", a joke due to Prof. Katayama